

Fundamental Trigonometric Identities

Reciprocal Identities

$$\begin{aligned} \sin x &= \frac{1}{\csc x} & \cos x &= \frac{1}{\sec x} & \tan x &= \frac{1}{\cot x} \\ \csc x &= \frac{1}{\sin x} & \sec x &= \frac{1}{\cos x} & \cot x &= \frac{1}{\tan x} \end{aligned}$$

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \quad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \quad 1 + \tan^2 x = \sec^2 x \quad 1 + \cot^2 x = \csc^2 x$$

Even-Odd Identities

$$\begin{aligned} \sin(-x) &= -\sin x & \cos(-x) &= \cos x & \tan(-x) &= -\tan x \\ \csc(-x) &= -\csc x & \sec(-x) &= \sec x & \cot(-x) &= -\cot x \end{aligned}$$

$$\sin x \tan x + \cos x = \sec x$$

$$\begin{aligned} \sin x \tan x + \cos x &= \sin x \left(\frac{\sin x}{\cos x} \right) + \cos x && \text{Apply a quotient identity: } \tan x = \frac{\sin x}{\cos x} \\ &= \frac{\sin^2 x}{\cos x} + \frac{\cos x}{1} && \text{Multiply.} \\ &= \frac{\sin^2 x}{\cos x} + \cos x \cdot \frac{\cos x}{\cos x} && \text{The least common denominator is } \cos x. \\ &&& \text{Write the second expression with a denominator of } \cos x. \\ &= \frac{\sin^2 x}{\cos x} + \frac{\cos^2 x}{\cos x} && \text{Multiply.} \\ &= \frac{\sin^2 x + \cos^2 x}{\cos x} && \text{Add numerators and place this sum over the least common denominator.} \\ &= \frac{1}{\cos x} && \text{Apply a Pythagorean identity: } \sin^2 x + \cos^2 x = 1. \\ &= \sec x && \text{Apply a reciprocal identity: } \sec x = \frac{1}{\cos x}. \end{aligned}$$

$$\begin{aligned}
 \cos x - \cos x \sin^2 x &= \cos x(1 - \sin^2 x) \\
 &= \cos x \cdot \cos^2 x \\
 &= \cos^3 x
 \end{aligned}$$

Factor $\cos x$ from the two terms.

Use a variation of $\sin^2 x + \cos^2 x = 1$.

Solving for $\cos^2 x$, we obtain

$$\cos^2 x = 1 - \sin^2 x.$$

Multiply.

$$\begin{aligned}
 \frac{1 + \sin \theta}{\cos \theta} &= \frac{1}{\cos \theta} + \frac{\sin \theta}{\cos \theta} \\
 &= \sec \theta + \tan \theta
 \end{aligned}$$

Divide each term in the numerator by $\cos \theta$.

Apply a reciprocal identity and a quotient identity:

$$\sec \theta = \frac{1}{\cos \theta} \text{ and } \tan \theta = \frac{\sin \theta}{\cos \theta}.$$

$$\begin{aligned}
 \sec \theta + \tan \theta &= \frac{1}{\cos \theta} + \frac{\sin \theta}{\cos \theta} \\
 &= \frac{1 + \sin \theta}{\cos \theta}
 \end{aligned}$$

Apply a reciprocal identity and a quotient identity:

$$\sec \theta = \frac{1}{\cos \theta} \text{ and } \tan \theta = \frac{\sin \theta}{\cos \theta}.$$

Add numerators. Put this sum over the common denominator.

$$\begin{aligned}
& \frac{\cos x}{1 + \sin x} + \frac{1 + \sin x}{\cos x} \\
&= \frac{\cos x(\cos x)}{(1 + \sin x)(\cos x)} + \frac{(1 + \sin x)(1 + \sin x)}{(1 + \sin x)(\cos x)} \\
&= \frac{\cos^2 x}{(1 + \sin x)(\cos x)} + \frac{1 + 2 \sin x + \sin^2 x}{(1 + \sin x)(\cos x)} \\
&= \frac{\cos^2 x + 1 + 2 \sin x + \sin^2 x}{(1 + \sin x)(\cos x)} \\
&= \frac{(\sin^2 x + \cos^2 x) + 1 + 2 \sin x}{(1 + \sin x)(\cos x)} \\
&= \frac{1 + 1 + 2 \sin x}{(1 + \sin x)(\cos x)} \\
&= \frac{2 + 2 \sin x}{(1 + \sin x)(\cos x)} \\
&= \frac{2(1 + \sin x)}{(1 + \sin x)(\cos x)} \\
&= \frac{2}{\cos x} \\
&= 2 \sec x
\end{aligned}$$

The least common denominator is $(1 + \sin x)(\cos x)$.

Rewrite each fraction with the least common denominator.

Use the FOIL method to multiply $(1 + \sin x)(1 + \sin x)$.

Add numerators. Put this sum over the least common denominator.

Regroup terms to apply a Pythagorean identity.

Apply a Pythagorean identity:
 $\sin^2 x + \cos^2 x = 1$.

Add constant terms in the numerator: $1 + 1 = 2$.

Factor and simplify.

Apply a reciprocal identity:

$$\sec x = \frac{1}{\cos x}$$

$$\begin{aligned}
\frac{\sin x}{1 + \cos x} &= \frac{\sin x}{1 + \cos x} \cdot \frac{1 - \cos x}{1 - \cos x} \\
&= \frac{\sin x(1 - \cos x)}{1 - \cos^2 x} \\
&= \frac{\sin x(1 - \cos x)}{\sin^2 x} \\
&= \frac{1 - \cos x}{\sin x}
\end{aligned}$$

Multiply numerator and denominator by $1 - \cos x$.

Multiply. Use $(A + B)(A - B) = A^2 - B^2$, with $A = 1$ and $B = \cos x$, to multiply denominators.

Use a variation of $\sin^2 x + \cos^2 x = 1$. Solving for $\sin^2 x$, we obtain $\sin^2 x = 1 - \cos^2 x$.

Simplify: $\frac{\sin x}{\sin^2 x} = \frac{\cancel{\sin x}}{\cancel{\sin x} \cdot \sin x} = \frac{1}{\sin x}$

$$\frac{\tan x - \sin(-x)}{1 + \cos x} = \frac{\tan x - (-\sin x)}{1 + \cos x}$$

$$= \frac{\tan x + \sin x}{1 + \cos x}$$

$$= \frac{\frac{\sin x}{\cos x} + \sin x}{1 + \cos x}$$

$$= \frac{\frac{\sin x}{\cos x} + \frac{\sin x \cos x}{\cos x}}{1 + \cos x}$$

$$= \frac{\frac{\sin x + \sin x \cos x}{\cos x}}{1 + \cos x}$$

$$= \frac{\sin x + \sin x \cos x}{\cos x} \div \frac{1 + \cos x}{1}$$

$$= \frac{\sin x + \sin x \cos x}{\cos x} \cdot \frac{1}{1 + \cos x}$$

$$= \frac{\sin x \cancel{(1 + \cos x)}}{\cos x} \cdot \frac{1}{\cancel{1 + \cos x}}$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x$$

The sine function is odd:
 $\sin(-x) = -\sin x$.

Simplify.

Apply a quotient identity:

$$\tan x = \frac{\sin x}{\cos x}$$

Express the terms in the numerator with the least common denominator, $\cos x$.

Add in the numerator.

Rewrite the main fraction bar as \div .

Invert the divisor and multiply.

Factor and simplify.

Multiply the remaining factors in the numerator and in the denominator.

Apply a quotient identity.

Verify the identity: $\frac{1}{1 + \cos \theta} + \frac{1}{1 - \cos \theta} = 2 + 2 \cot^2 \theta$.

Solution We begin by working with the left side.

$$\frac{1}{1 + \cos \theta} + \frac{1}{1 - \cos \theta}$$

$$= \frac{1(1 - \cos \theta)}{(1 + \cos \theta)(1 - \cos \theta)} + \frac{1(1 + \cos \theta)}{(1 + \cos \theta)(1 - \cos \theta)}$$

$$= \frac{1 - \cos \theta + 1 + \cos \theta}{(1 + \cos \theta)(1 - \cos \theta)}$$

$$= \frac{2}{(1 + \cos \theta)(1 - \cos \theta)}$$

$$= \frac{2}{1 - \cos^2 \theta}$$

The least common denominator is $(1 + \cos \theta)(1 - \cos \theta)$.

Rewrite each fraction with the least common denominator.

Add numerators. Put this sum over the least common denominator.

Simplify the numerator: $-\cos \theta + \cos \theta = 0$ and $1 + 1 = 2$.

Multiply the factors in the denominator.

$$2 + 2 \cot^2 \theta = 2 + 2 \left(\frac{\cos^2 \theta}{\sin^2 \theta} \right)$$

$$= \frac{2 \sin^2 \theta}{\sin^2 \theta} + \frac{2 \cos^2 \theta}{\sin^2 \theta}$$

$$= \frac{2 \sin^2 \theta + 2 \cos^2 \theta}{\sin^2 \theta}$$

$$= \frac{2(\sin^2 \theta + \cos^2 \theta)}{\sin^2 \theta}$$

$$= \frac{2}{\sin^2 \theta}$$

$$= \frac{2}{1 - \cos^2 \theta}$$

Use a quotient identity: $\cot \theta = \frac{\cos \theta}{\sin \theta}$.

Rewrite each term with the least common denominator, $\sin^2 \theta$.

Add numerators. Put this sum over the least common denominator.

Factor out the greatest common factor, 2.

Apply a Pythagorean identity: $\sin^2 \theta + \cos^2 \theta = 1$.

Use a variation of $\sin^2 \theta + \cos^2 \theta = 1$ and solve for $\sin^2 \theta$: $\sin^2 \theta = 1 - \cos^2 \theta$.

$$\begin{aligned}
 \text{a. } \ln|\sec x| &= \ln\left|\frac{1}{\cos x}\right| & \sec x &= \frac{1}{\cos x} \\
 &= \ln 1 - \ln|\cos x| & \ln\left(\frac{M}{N}\right) &= \ln M - \ln N \\
 &= -\ln|\cos x| & \ln 1 &= 0
 \end{aligned}$$

Verify the identity.

$$1 + \frac{\tan^2 \theta}{\sec \theta + 1} = \sec \theta \Rightarrow 1 + \frac{\sec^2 \theta - 1}{\sec \theta + 1}$$

$$1 + \frac{\tan^2 \theta}{\sec \theta + 1} = \sec \theta$$

$$1 + \frac{\overbrace{1 + \tan^2 \theta}^{\sec^2 \theta}}{\sec \theta + 1} = \sec \theta$$

$$1 + \frac{\sec^2 \theta - 1}{\sec \theta + 1} = \sec \theta$$

$$1 + \frac{(\cancel{\sec \theta + 1})(\sec \theta - 1)}{(\cancel{\sec \theta + 1})} = \sec \theta$$

$$\cancel{1 + \sec \theta} - \cancel{1}$$

$$\sec \theta = \sec \theta$$

$$a^2 - b^2 = (a - b)(a + b)$$

$$(a + b)^2$$

$$(a + b)(a + b)$$

$$a^2 + ab + ab + b^2$$

$$a^2 + 2ab + b^2$$

Use properties of natural logarithms and fundamental trigonometric identities to show that the pair of expressions is equivalent.

$$\ln |1 + \cos \theta| - 2 \ln |\sin \theta| \text{ and } -\ln |1 - \cos \theta|$$

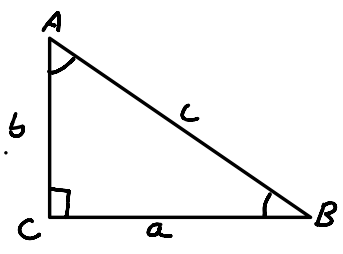
Graph two periods of the given secant function.

$$y = 4 \sec \left(x + \frac{5\pi}{4} \right)$$

OPPOSITE side
does NOT Touch angle

adj Side Touches
angle, NOT Hyp

Hyp is Longest
side



$$\sin A = \frac{a}{c}$$

$$\cos A = \frac{b}{c}$$

$$\tan A = \frac{a}{b}$$

$$\sin B = \frac{b}{c}$$

$$\cos B = \frac{a}{c}$$

$$\tan B = \frac{b}{a}$$

A forest ranger sights a fire directly to the south. A second ranger, 9 miles east of the first ranger, also sights the fire. The bearing from the second ranger to the fire is S 32° W. How far is the first ranger from the fire?



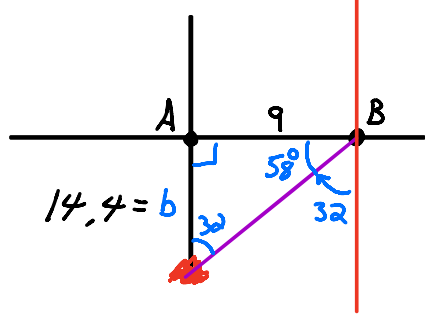
How far is the first ranger from the fire?

mi (Round to the nearest tenth of a mile.)

$$\tan 58 = \frac{b}{9}$$

$$9 \cdot 1.6003 = \frac{b}{9} \cdot 9$$

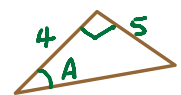
$$14.4 = b$$



A jet leaves a runway whose bearing is N 38° E from the control tower. After flying 4 miles, the jet turns 90° and flies on a bearing of S 52° E for 5 miles. At that time, what is the bearing of the jet from the control tower?



The jet's bearing from the control tower is N 89° E.
(Round to the nearest degree as needed.)



$$\tan A = \frac{5}{4} = 1.25$$

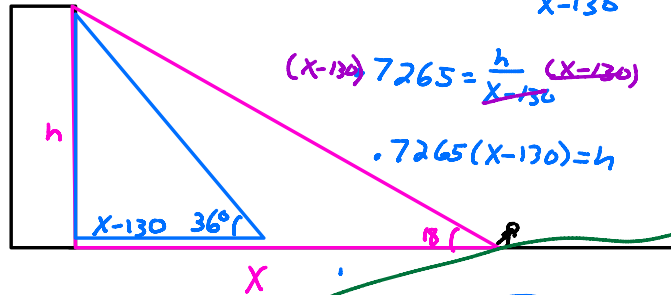
$$A = \tan^{-1} 1.25$$

$$A = 51.34$$

N 89.34 E

The angle of elevation to the top of a building changes from 18° to 36° as an observer advances 130 feet toward the building. Find the height of the building, x , to the nearest foot.

The height of the building is approximately feet.
 (Do not round until the final answer. Then round to the nearest whole number as needed.)



$$\tan 36 = \frac{h}{x-130}$$

$$\tan 18 = \frac{h}{x}$$

$$(x-130) \cdot 7265 = \frac{h}{x-130} \cdot (x-130)$$

$$.3249 = \frac{h}{x}$$

$$.7265(x-130) = h$$

$$.3249x = h$$

$$.7265(x-130) = .3249x$$

$$0.7265x - 94.45 = .3249x$$

$$-0.7265x \quad -0.7265x$$

$$-94.45 = -0.4016x$$

$$-0.4016 \quad -0.4016$$

$$x = 235.18$$

$$h = 0.3249(235.18)$$

$$h = 76.41$$

Solve the right triangle shown in the figure.

$$B = 18.8^\circ, b = 31.5$$

$$\sin B = \frac{b}{c}$$

$$\sin 18.8 = \frac{31.5}{c}$$

$$c \cdot .322266 = \frac{31.5}{c}$$

$$\frac{.322266c}{.322266} = \frac{31.5}{.322266} = 97.745$$

$$A = 71.2^\circ \text{ (Round to the nearest tenth as needed.)}$$

$$a \approx 92.53 \text{ (Round to the nearest hundredth as needed.)}$$

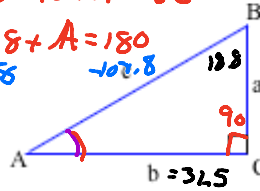
$$c \approx 97.75 \text{ (Round to the nearest hundredth as needed.)}$$

$$18.8 + 90 + A = 180$$

$$108.8 + A = 180$$

$$-108.8 \quad -108.8$$

$$A = 71.2$$



$$\sin A = \frac{a}{c}$$

$$\cos B = \frac{c}{a}$$

$$\cos 18.8 = \frac{a}{97.745}$$

$$.946649 = \frac{a}{97.745}$$

$$97.745$$

$$92.530 = a$$

Solve the right triangle shown in the figure.

$a = 28.4, c = 51.9$

$\cos A = \frac{b}{c}$

$\cos 33.176 = \frac{b}{51.9}$

$51.9 \cdot 0.836998 = \frac{b}{51.9} \rightarrow 51.9$

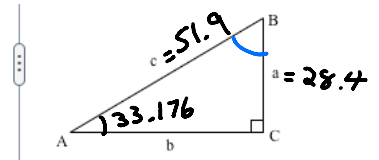
$43.44 = b$

$\sin A = \frac{28.4}{51.9}$

$\sin A = 0.5472062$

$A = \sin^{-1} 0.5472062$

$A = 33.1756$

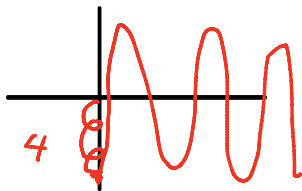


$90 + 33.176 + B = 180$

An object is attached to a coiled spring. The object is pulled down (negative direction from the rest position) 4 centimeters, and then released. Write an equation for the distance d of the object from its rest position, after t seconds if the amplitude is 4 centimeters and the period is 6 seconds.

The equation for the distance d of the object from its rest position is $d = -4 \cos\left(\frac{\pi}{3}t\right)$.

(Type an exact answer, using π as needed. Use integers or fractions for any numbers in the equation.)



$d = -4 \cos Bx$

$d = -4 \cos \frac{\pi}{3}x$

Period = 6 sec

Normal period = 2π

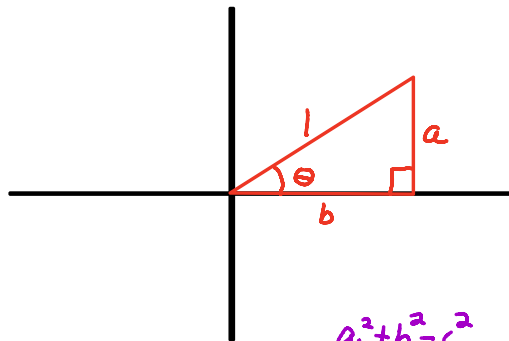
$\cos Bx$ To Find Period, set $Bx = 2\pi$

Solve For x

x is new period

$Bx = 2\pi$
 $\frac{B(6) = 2\pi}{6 \quad 6}$

$B = \frac{\pi}{3}$



$$\sin \theta = \frac{a}{1} = a$$

$$\cos \theta = \frac{b}{1} = b$$

$$\sin \theta = a$$

$$\cos \theta = b$$

$$a^2 + b^2 = c^2$$

$$a^2 + b^2 = 1^2 = 1$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\frac{\sin \theta}{\cos \theta} = \tan \theta$$

$$\frac{1}{\cos \theta} = \sec \theta$$

$$\frac{\cos \theta}{\sin \theta} = \cot \theta = \frac{1}{\tan \theta}$$

$$\frac{1}{\sin \theta} = \csc \theta$$

$$\frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

$$\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta}$$

$$\frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta}$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

$$(\csc x - \cot x)^2$$

$$\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right)^2 =$$

$$\frac{(1 - \cos x)^2}{\sin^2 x}$$

$$\frac{(1 - \cos x)^2}{1 - \cos^2 x}$$

$$\frac{\cancel{(1 - \cos x)}(1 - \cos x)}{\cancel{(1 - \cos x)}(1 + \cos x)}$$